Note on Linear Programming

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1 What is Linear Programming

1.1 Standard LP

Definition 1.1 (Standard LP)

$$\min c^T x$$
$$s.t. Ax \leq b, x \geq 0 \quad (1)$$

Note on Perspective of geometry *c determines the direction of fastest increase, cx denote the hyperplane with the direction c of fastest increase, and A define the feasible region by clarifying the intersection of half spaces.*

Note on Example of standard LP transformation

1. *"Max" to "Min": Adding negative sign to the objective.*
2. $a_{i1}x_1 + \cdots + a_{in}x_n \geqslant b_i \Rightarrow a_{i1}x_1 + \cdots + a_{in}x_n - x_{n+1} = b_i$, *where x_{n+1} is the surplus variable.*
3. $a_{i1}x_1 + \cdots + a_{in}x_n \leqslant b_i \Rightarrow a_{i1}x_1 + \cdots + a_{in}x_n + x_{n+1} = b_i$, *where x_{n+1} is the slack variable.*
4. *Free variable:* $x_i = x_i^+ - x_i^-, x_i^+ \geqslant 0, x_i^- \geqslant 0$
5. *Absolute variable:* $|x_j| = x_j^+ + x_j^-, x_j = x_j^+ - x_j^-$, *adding a constraint* $x_j^+ \cdot x_j^- = 0$, *note that this constraint can be neglected if* $c_j \geq 0$
6. a^+ *or* $\max(a, 0)$*:* $\max 170(3x-240)^+ - 238(240-3x)^+$ *to the following LP. If* $3x-240 > 0$, y_1 *increases to* $3x - 240$ *and* y_2 *decreases to 0 (*$y_1 > y_2$*), If* $3x - 240 < 0$, y_1 *increases to 0,* y_2 *decreases to* $240 - 3x$ *(*$y_1 < y_2$*).*

$$\max 170y_1 - 238y_2$$
$$s.t. 3x - 240 = y_1 - y_2, y_1, y_2 \geq 0 \quad (2)$$

7. *Quantity discount: For example, if* $p = 4000$ *for* $x < 30$, $p = 2000$ *for* $30 \leq x < 50$ *and* $p = 1500$ *for* $x > 50$. *Using two binary variables:* σ_2, σ_3. *If* $\sigma_2, \sigma_3 = 0, 0$, *it means that* $x_2 = x_3 = 0$. *And* $\sigma_2, \sigma_3 = 0, 1$ *does not exist. If* $\sigma_2, \sigma_3 = 1, 0$, *it means that*

$0 \le x \le 20, x_3 = 0$. *If* $\sigma_2, \sigma_3 = 1, 1$, *it means that* $0 \le x \le 20, x_3 \le M$.

$$\begin{aligned} &\min 4000x_1 + 2000x_2 + 1500x_3 \\ &s.t. x_1 + x_2 + x_3 = Demand \\ &x_2 \le 20\sigma_2 \\ &x_1 \ge 30\sigma_2 \\ &x_2 \ge 20\sigma_3 \\ &x_3 \le M\sigma_3 \end{aligned} \tag{3}$$

8. $b_i < 0$: *Adding negative sign to the whole constraints.*
9. $x \le 0$: *Let* $x' = -x$
10. $l \le x \le u$

$$\begin{aligned} &\min c^T x && \min c^T x^+ - c^T x^- \\ &s.t. Ax \le b && s.t. Ax^+ - Ax^- + s_1 = b \\ &l \le x \le u && x^+ - x^- + s_2 = u \\ & && x^+ - x^- - s_3 = l \\ & && x^+, x^-, s_1, s_2, s_3 \ge 0 \end{aligned} \tag{4}$$

11. *Linear Fractional Programming: Define* $y = \frac{x}{e^T x + f}$ *and* $z = \frac{1}{e^T x + f}$, *here* z *in LP1 cannot be zero, though* z *in LP2 can be zero, we can show that these two are equivalent. (1)* z^* *in LP2 is not zero, then the optimal solution are the same; (2)* z^* *in LP2 is zero, then it means* $e^T x^* + f \to \infty$.

$$\begin{aligned} & \textbf{LP1} && \textbf{LP2} \\ \max_x \; & \frac{c^T x + d}{e^T x + f} & \min_{y,z} \; & c^T y + dz \\ s.t. \; & Ax \le b & s.t. \; & Ay - bz \le 0 \\ & e^T x + f > 0 & & e^T y + fz = 1 \\ & & & z \ge 0 \end{aligned} \tag{5}$$

1.2 Basic and Optimal Solution

Definition 1.2 (Feasible, Basic, Optimal, Degenerate Solution)

1. *Feasible solution:= a solution which satisfies the constraints* $Ax = b, x \ge 0$.
2. *Basic solution:=* $x = (x_B, x_N)$, *where* x_B *is linear independent* $m \times m$ *matrix,* x_N *is* $m \times n - m$ *matrix, the solution attained by set* x_N *to zero.*
3. *Basic feasible solution:= A solution which is feasible and basic.*
4. *Degenerate basic solution:= A basic solution with one or more basic variables has the value zero.*
5. *Optimal feasible solution:= A feasible solution that achieves the minimum value.*

6. *Optimal basic feasible solution:= A optimal feasible solution which is also basic.*

Note on Geometric Interpretation of Degenerate Solution *In the two-dimensional space, degenerate solution denotes the intersection of three or more lines. In the three-dimensional space, degenerate solution denotes the intersection of four or more planes. The nature of degenerate solution is that it remains the same point after pivoting.*

Note on *Several optimal solutions does not mean there exist at least two basic feasible solution that are optimal, e.g.* $\{(x, y) \in \mathbb{R}^2 \mid -x + y = 0, \quad x, y \geq 0\}$. *Only one basic feasible solution, but the whole line is optimal.*

1.3 LP with Bounded Variables

$$\begin{aligned} \min c^T x \qquad & \Rightarrow c^T(x_B, x_N) \\ \text{s.t.} Ax = b \qquad & \Rightarrow I x_B + \bar{A} x_N = \bar{b} \\ l \leq x \leq u & \end{aligned} \tag{6}$$

Definition 1.3 (Generalized definition and condition)

- *Basic solution* $:= x_N$ *equal to either the lower bound or upper bound.*
- *Degenerate basic solution* $:=$ *one or more* $x_B = l$ *or* u*.*
- *Optimality condition* $:=$ *A basic solution* $x = (x_B^*, x_N^*)$ *is optimal if*
  - $l_B \leq x_B^* \leq u_B$*. (feasibility)*
  - $r_j \geq 0 \quad \forall j \in L = \{j \in N | x_j^* = l_j\}$ *and* $r_j \leq 0 \quad \forall j \in U = \{j \in N | x_j^* = u_j\}$.

2 Feasibility

Definition 2.1 (Feasible Direction)

Let x *be an element of a polyhedron* P*. A vector* $d \in R^n$ *is said to be a feasible direction at* x*, if there exists a positive scalar* θ *for which* $x + \theta d \in P$*.*

Lemma 2.1 (Feasible Direction (Bertsimas et al., 1997, P. 129))

For polyhedron $P = \{x \in \Re^n \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$*, a vector* $d \in R^n$ *is a feasible direction at* x *iff* $Ad = 0$ *and* $d_i \geq 0$ *for every* i *such that* $x_i = 0$*.*

3 Optimality and Uniqueness

Proposition 3.1 (Interior point and Optimality)

$$\min c^T x$$
$$s.t. Ax \leq b, x \geq 0 \tag{7}$$

The point satisfies that $Ax_0 < b, x_0 > 0$ *cannot be an optimal solution.*

Proof Suppose for the sake of contradiction that there exists another point $x_0 \pm \varepsilon c > 0$ and $A(x_0 \pm \varepsilon c) < b$, then we show that $c^T(x_0 \pm \varepsilon c) = c^T x_0 \pm \varepsilon ||c||$, that is, the new point is more optimal than the former one.

Next we construct $\varepsilon > 0$ that we want, for any $x_0 + \varepsilon c > 0$,x_0, $x_{0i} > 0$ must hold. However, $c_i < 0$ may occur, we let $\varepsilon < \min\{\frac{x_i}{|c_i|}\} \forall c_i < 0$. For $A(x_0 + \varepsilon c) < b$, in each row, we want $\sum_{i=1}^n a_i(x_i + \varepsilon c_i) < b_i$. Thus, we can let $\varepsilon < \min\{\frac{b_i - \sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i c_i}\}$. ■

Theorem 3.1 (Optimality Conditions (Bertsimas et al., 1997, P. 129))

Consider the problem of minimizing $c^T x$ *over a polyhedron* P

1. *A feasible solution* x *is optimal iff* $c^T d \geq 0$ *for every feasible direction* d *at* x
2. *A feasible solution* x *is unique optimal iff* $c^T d > 0$ *for every nonzero feasible direction* d *at* x

Theorem 3.2 (Conditions for a unique optimum (Bertsimas et al., 1997, P. 129))

Let X *be a basic feasible solution with basis* B

1. *If the reduced cost of every nonbasic variable is positive, then* x *is the unique optimal solution.*
2. *If* x *is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.*

4 LP based on Basis and Topological Space

4.1 Caratheodory's theorem

Proposition 4.1 (Caratheodory's theorem)

Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *be a collection of vectors in* R^m, *Let*

$$C = \left\{ \sum_{i=1}^n \lambda_i \mathbf{A}_i \mid \lambda_1, \ldots, \lambda_n \geq 0 \right\}$$

Then any element of C *can be expressed in the form* $\sum_{i=1}^n \lambda_i \mathbf{A}_i$, *with* $\lambda_i \geq 0$, *and with at most* m *of the coefficients* λ_i *being nonzero.*

Proof When $n \leq m$, obviously the condition holds. When $n > m$, consider a polyhedron

$$\Lambda = \left\{ (\lambda_1, \ldots, \lambda_n) \in \Re^n \mid \sum_{i=1}^{n} \lambda_i \mathbf{A}_i = \mathbf{y}, \lambda_1, \ldots, \lambda_n \geq 0 \right\}$$

This is a standard LP, thus there is at least a extreme point, that is, a basic feasible solution $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Note that we have at most m linear independent vectors among A_i, thus a basic feasible solution has at least $n - m$ zero components, which means there are at most m non-zero components in λ^*. ■

4.2 Feasibility to Basic Feasibility

Theorem 4.1 (Fundamental theorem of Linear Programming)

Given a LP in standard form, where A is a $m \times n$ matrix of rank m:

- *If there is a feasible solution, then there is a basic feasible solution.*
- *If there is an optimal feasible solution, then there is a basic optimal feasible solution.*

Remark That is, feasibility must lead to basic feasibility.

Proof [1] A feasible solution $\Longleftrightarrow$ constraint set is not empty $\Longleftrightarrow$ polytope is not empty, thus theorem also can be interpreted as feasibility must lead to basic feasibility. Let x be a feasible solution, then $Ax = b$ and $x \geq 0$. That is, $a_1x_1 + \cdots + a_nx_n = b$ to $a_1x_1 + \cdots + a_kx_k = b$ (some of x_i is zero). There are two possible cases:

1. $a_1, ..., a_k$ are LIN
2. $a_1, ..., a_k$ are not LIN

(i) If $k = m$, then x is a basic solution, Done! If $k < m$, since $A_{m\times n}$ is full rank, then $a_1x_1 + \cdots + a_kx_k + a_{k+1}0 + \cdots + a_m \cdot 0 = b$ is a basic solution, Done!

(ii) We can find the following equations, let (2)-ε(1)=$a_1(x_1 - \varepsilon y_1) + \cdots + a_k(x_k - \varepsilon y_k) = b$, where $\varepsilon > 0$. And this is another solution to this LP, As ε increases, some of $x_i - \varepsilon y_i$ go down to zero. Repete it, we can get LIN $a_1', \cdots, a_k'$. Note that y_i can be positive or negative, however, $x_i - \varepsilon y_i$ must be positive when ε is small enough. And some of $x_i - \varepsilon y_i$ goes closer to 0 when ε increases.

$$\begin{cases} a_1y_1 + \cdots + a_ky_k = 0 & (1) \\ a_1x_1 + \cdots + a_kx_k = b & (2) \end{cases}$$

■

Proof [2] This is equal to show that if x is optimal, then $x - \varepsilon y$ is optimal. When ε is small, $x - \varepsilon y > 0$ then feasible, $c^\top(x - \varepsilon y) = c^\top x - \varepsilon c^\top y < c^\top x$ if $\sum c^\top y > 0$ (we can choose the sign of ε arbitrarily). Since x is optimal feasible solution, $c^T x$ is the minimal, $c^T y$ must equal to zero. Then $x - \varepsilon y$ is a optimal solution too, and by choosing ε we can get a optimal basic solution. ■

4.3 Basic Feasible Solution and Extreme Point

Theorem 4.2 (Extreme Point = Basic feasible solution)

Let K be the convex polytope of $H = \{Ax = b : x \in R^n, x \geqslant 0\}$. A vector x is an extreme point of K iff x is a basic feasible solution to H.

Proof If side, to show, if x is a basic feasible solution, then x is an extreme point. x is feasible $\Longleftrightarrow x = (x_1, \ldots, x_m, 0, \ldots, 0)$ where $x_i \geq 0$ and $a_1x_1 + \cdots + a_mx_m = b$. Suppose $y = (y_1, \ldots, y_m, ; y_{m+1}, \ldots, y_n), z = (n_1, \cdots, z_m; z_{m+1}, \ldots, z_n)$ are other solutions in H. Suppose for the sake of contradiction that $\exists \alpha \in (0, 1), \alpha y + (1 - \alpha)z = x$, i.e., x can be represented by y and z. Then we have $\alpha y_i + (1 - \alpha)z_i = 0$ for all $i = m + 1, ..., n$, it means $y_i = z_i = 0 \forall i = m + 1, ..., n$. Since $a_1, ..., a_m$ are LIN, there is only one kind of representation, $x_i = y_i = z_i, i = 1, ..., m$. Done!

Only if side: Say x is a extreme point. Then (1) $a_1x_1 + \cdots + a_kx_k = b$ (since x is a feasible solution). We want to prove that $a_1, ..., a_k$ are LIN. Assume that $a_1, ..., a_k$ are not LIN. We can find (2) $a_1y_1 + \cdots + a_ky_k = 0$. Construct the following equations, then $x = \frac{1}{2}(\hat{x} + \bar{x})$, that is, x is not an extreme point. Thus $a_1, ..., a_k$ must be LIN. Done!

$$\begin{cases} \hat{x} = x + \varepsilon y & (1) + \varepsilon(2) \\ \overline{x} = x - \varepsilon y & (1) - \varepsilon(2) \end{cases}$$

■

Corollary 4.1 (Nonempty Standard LP always has an extreme point)

If the convex set K corresponding to $\{Ax = b, x \geqslant 0\}$ is non-empty, then it has at least one extreme point.

Remark However, this does not mean that every nonempty polyhedron has at least one extreme point, e.g., the half space $\{(x, y) \in \mathbb{R}^2 \mid x + y \geq 1\}$.

Corollary 4.2 (Optimality and Extreme Point)

If there is a feasible solution that is optimal to a LP, then there is an optimal finite solution that is an extreme point of the constraint set.

Note on *Note that this corollary does not clarify that optimal solution is exactly the extreme point, since this optimal solution may be at the middle of the optimal line.*

Corollary 4.3 (Feasible Region and Finite Extreme Point)

The constraint set K corresponding to $\{Ax = b, x \geqslant 0\}$ has at most a finite number of extreme points.

Note on Finite *From the perspective of combination, there are at most* $\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!}$ *extreme points.*

Note on *These three corollary connects solution and extreme point via constraint set.*

4.4 Degeneracy

Definition 4.1 (Degenerate)

A basic solution $x \in R^n$ is said to be degenerate if more than n of the constraints are active at x, e.g. $a_i x = b_i$.

Definition 4.2 (Degeneracy in Standard form)

Consider the standard form polyhedron $P = \{\mathrm{x} \in \Re^n \mid \mathbf{Ax} = \mathbf{b}, x \geq 0\}$ and let x be a basic solution. Let m be the number of rows of A. The vector x is a degenerate basic solution if more than $n - m$ of the components of x are zero.

Lemma 4.1

If two different bases lead to the same basic solution, then this basic solution is degenerate, but not vice versa.

Proof Assume not degenerate, then the basic solution have $n - m$ zero components, this uniquely determine m non-zero components, which correspond to a unique choice of basis. Contradiction.

Counterexample is $\left\{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0, x - y \geq 0 \quad x, y \geq 0\right\}$, this polyhedron contains only one degenerate point $(0, 0)$, but there is only one choice of basis. ∎

Lemma 4.2

For degenerate solution, there is possible for non-basic variable's reduced cost to be negative, while it is still a optimal solution.

Proof Counterexample is $\left\{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0, x - y \geq 0 \quad x, y \geq 0\right\}$. ∎

5 Simplex Method

Assumption 5.1 (Non-degeneracy Assumption)

Every basic feasible solution is not degenerate, that is, $x_i > 0, \ i = 1, \ldots, m$ *for* $(x_1, \ldots, x_m, 0, \ldots, 0)$.

5.1 Pivoting: From basis to basis

Definition 5.1 (Pivoting)

Pivoting ⇔ basis change ⇔ one extreme point to another ⇔ one basic solution to another. Note that the basic solution obtained by pivoting may not be feasible (may negative).

Suppose we have a modified A' in this stage, where y_{i0} is b, and we want to do pivoting based on A'.

$$A' = \begin{pmatrix} 1 & 0 & \dots & 0 & y_{1,m+1} & \dots & y_{1,n} & y_{1,0} \\ 0 & 1 & \dots & 0 & y_{2,m+1} & \dots & y_{2,n} & y_{2,0} \\ \dots & & \dots & & \dots & & \dots & \\ 0 & 0 & \dots & 1 & y_{m,m+1} & \dots & y_{m,n} & y_{m,0} \end{pmatrix}$$

Interpretation of Pivoting From the Perspective of Row: Suppose we want to use x_q $(m+1 \le q \le n)$ to replace x_p $(1 \le p \le m)$, pivoting is exactly

- Row p divided by $y_{p,q}$ (only when $y_{p,q} \neq 0$).
- Rows except Row p minus Row p and times $y_{i,q}$:

$$y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, \quad i \neq p; \quad y'_{pj} = \frac{y_{pj}}{y_{pq}}$$

Interpretation of Pivoting From the Perspective of Column: The polytope $\{Ax = b, x \geqslant 0\}$ can be perceived as a linear combination of $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. As for the canonical form, $a_j = y_{1j}a_1 + \cdots + y_{nj}a_n$ $\forall j = m+1, \dots, n$ and $b = y_{10}a_1 + y_{20}a_2 + \cdots + y_{m0}a_m$. Suppose we want to use x_q $(m+1 \le q \le n)$ to replace x_p $(1 \le p \le m)$, pivoting is exactly

- With $a_q = y_{pq}a_p + \sum_{i=1,i\neq p}^{m} y_{iq}a_i$, solve for a_p, we know $a_p = \frac{1}{y_{pq}}a_q - \sum_{i=1,i\neq p}^{m} \frac{y_{iq}}{y_{pq}} a_i$.
- By substitution with a_p, we know $a_j = \frac{y_{pj}}{y_{pq}}a_q + \sum_{i=1,i\neq p}^{m} \left(y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \right) a_i, \quad j = m+1, \dots, n, j \neq q$.
- That is, we do a transformation as follows

$$\begin{cases} y'_{ij} = y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj}, i \neq p \\ y'_{pj} = \frac{y_{pj}}{y_{pq}} \end{cases}$$

5.2 Entering basic variable

There are two ways to select the entering variable, 1st way is selecting the variable with the most negative reduced cost. However, 2nd way may be a better criterion, if we select the variable which, when pivoted in, will produce the greatest improvement in the objective function, that is, select the variable x_k corresponding to the index k that minimizes $\underset{i, y_{k,i}>0}{\text{Max}} \{r_k \cdot y_{0,i}/y_{k,i}\}$, here $\mathbf{x}_B = \mathbf{y}_0 = \mathbf{B}^{-1}\mathbf{b}$ is the current basic solution, $\mathbf{r}^T = \mathbf{c}^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{A}$ is the reduced cost vector, and $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$ where a_k is the k^{th} column in A.

Proof [2nd way] Note that $z = z_0 + \sum_{i=m+1}^{n} r_i \cdot x_i$, and to min $\sum_{i=m+1}^{n} r_i \cdot x_i$, since we can only choose one, it is equal to min $\{r_i x_i\}$.

To maintain the feasibility, assume we want to increase x_k from 0 to ε for any $k = m+1, ..., n$, and we should hold $a_1(x_1 - \varepsilon y_{ik}) + \cdots + a_m(x_m - \varepsilon y_{ik}) + a_k\varepsilon = b$ and $x_i - \varepsilon y_{ik} \geqslant 0$ $\forall i = 1, ..., m$. Thus $\varepsilon = \min_{i=1,...,m}\{y_{i0}/y_{ik}\}$. And the whole problem is equal to $\min_k\{r_i \min_i\{y_{i0}/y_{ik}\}\}$, since $r_i < 0$, we can rewrite as $\min_{k=m+1,...,n} \max_i\{r_i \frac{y_{i0}}{y_{ik}}\}$. ■

5.3 Leaving basic variable

Suppose the basic solution is non-degenerate and $x_i > 0, \quad i = 1, \ldots, m$, if we choose $a_q, q > m$ to enter the basis, let $\varepsilon > 0$, then we have

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$$\Downarrow -\varepsilon \cdot (y_{1q}a_1 + y_{2q}a_2 + \cdots + y_{mq}a_m - a_q = 0) \tag{8}$$

$$a_1(x_1 - \varepsilon y_{1q}) + a_2(x_2 - \varepsilon y_{2q}) + \cdots + a_m(x_m - \varepsilon_{mq}) + \varepsilon a_q = b$$

Note that $(x_i - \varepsilon y_{iq})$ is a feasible solution as long as all $x_i - \varepsilon y_{iq} > 0$, but not a basic solution.

- If all $y_{iq} < 0$, then as ε increase, all $(x_i - \varepsilon y_{ia}) > 0$, this is a unbounded LP problem.
- If some $y_{iq} > 0$, as ε increase, we have $x_i - \varepsilon y_{iq} = 0$ for this variable, and we get a new basic solution. Let $\varepsilon_M = \min_i \left(\frac{x_i}{y_{iq}} : y_{iq} > 0 \right\}$, and the corresponding vector is the one we want.
- If more than one coefficient reduces to zero at $\varepsilon = \varepsilon_M$, the new basic solution is degenerate.

5.4 Optimality Test

Given a basic feasible solution $x = (x_1, \ldots, x_m, 0, \ldots, 0)$, let $z_0 = \sum_{i=1}^{m} c_i y_{i0}$ (i.e., current objective). Then the objective function can be rewritten as the type contains z_0, and whether there exists pivoting to optimize $\sum_{i=m+1}^{n} (c_i - z_i)\, x_i$ means whether the current solution is optimal.

$$\begin{aligned}
z &= \sum_{i=1}^{n} c_i x_i \\
&= \sum_{i=1}^{m} c_i x_i + \sum_{i=m+1}^{n} c_i x_i \\
&= \sum_{i=1}^{m} c_i (y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j) + \sum_{i=m+1}^{n} c_i x_i \qquad & (x_i = y_{i0} - \sum_{j=m+1}^{n} y_{ij} x_j) \\
&= z_0 + \sum_{i=m+1}^{n} (c_i - z_i)\, x_i \qquad & (z_i = \sum_{k=1}^{m} y_{kj} c_k)
\end{aligned}$$

Theorem 5.1 (Optimality Condition)

Given a non-degenerate basic feasible solution with corresponding objective function value z_0:

- *If $c_i - z_i < 0$ for some i, then there is a feasible solution with objective value $z < z_0$. If the column a_i can be substituted for some column in the original basis to yield a new basic feasible solution, then this new solution will have $z < z_0$. Otherwise if we found a vector d satisfying $Ad = 0, d \geq 0, c'd < 0$ (Bertsimas et al., 1997, P. 91), the constraint set is unbounded and the objective function value can be made arbitrarily small.*
- *If $c_i - z_i \geq 0$ for all i, then the solution is optimal.*

Lemma 5.1 (ε-optimal)

For a standard LP, say if $|z_0 - z^*| \leq \varepsilon$, then it would be enough, here z_0 is the current value of simplex. Let $\sum_i x_i \leq s$, then if $M = \max_j(z_j - c_j) \leq \varepsilon/s$, then $z_0 - z^* \leq \varepsilon$.

Proof Note that

$$\begin{aligned} |z - z_0| &= |\sum_{i=m+1}^{n} (c_i - z_i) x_i| \\ &\leq \sum_{i=m+1}^{n} |c_i - z_i| x_i && |\sum_i x_i| \leq \sum_i |x_i| \\ &\leq M \sum_{i=m+1}^{n} x_i && z_j - c_j \leq M \\ &\leq Ms \leq \varepsilon \end{aligned}$$

■

5.5 Matrix formulation of Simplex Method

Let $A = (B|D)$, $x = (x_B|x_D)$, and $c^T = (c_B^T, c_D^T)$. Formulating LP with matrix,

$$\begin{aligned} \text{Min} \quad & z = c_B^T x_B + C_D^T x_D \\ \text{s.t.} \quad & Bx_B + Dx_D = b \quad \Rightarrow \quad x_B = B^{-1}b - B^{-1}Dx_D \\ & x_B, x_D \geqslant 0 \end{aligned} \tag{9}$$

We have $z = c_B^\top B^{-1}b + \left(c_D{}^\top - C_B^\top B^{-1}D\right) x_D$, where $r_D{}^\top := c_D{}^\top - c_B{}^\top B^{-1}D$ is the reduced cost vector. We also have a basic solution $\left(x_B = B^{-1}b, x_D = 0\right)$.

- Initaialization: Given the basis B, the current solution is $B^{-1}b$.
- Step 1: Calculate the relative cost vector $r_D^T = c_D^T - c_B^T B^{-1} D$. If $r_D^T \geq 0$, the current basis is optimal. Otherwise, go to Step 2.
- Step 2: Determine which vector to enter the basis by selecting the most negative cost coefficient. Let it be column q and then $B^{-1}a_q$ gives the representation of a_q in terms of the vectors in the current basis B.
- Step 3: If all $y_{iq} \leq 0$, then stop and the problem is unbounded. Otherwise, calculate the ratio of $\frac{y_{i,0}}{y_{i,q}}$ for $y_{i,q} > 0$, and determine which variable to enter the basis.
- Step 4: Update B and the current solution $B^{-1}b$. Return to Step 1.

5.6 Simplex and Degeneracy

5.7 Modified Simplex Method for LP with Bounded Variables

- Given a feasible solution x^0.
- choose the entering variable be $s = \arg\min_{\{j \in L\} \cup \{k \in U\}} \{r_j, -r_k\}$, and define δ. We will

do the change to make x_s to $x_S = x_S^0 + \delta\theta, \theta \geq 0$.

$$\delta = \begin{cases} 1 & \text{if } x_S = l_S \\ -1 & \text{if } x_S = u_S \end{cases}$$

- choose the leaving variable: To maintain feasilibity, we need $l_S \leq x_S^0 + \delta\theta \leq u_S$ and $l_i \leq x_i^0 - \delta\theta y_{iS} \leq u_i, i = 1, 2 \ldots, m$, thus, $\theta = \min\{\theta_S, \theta_l, \theta_u\}$. Let $r = \operatorname{argmin}\{\theta_S, \theta_l, \theta_u\}$, x_r is the leaving variable we want.

$$\theta_S = u_S - l_S, \theta_l = \min_{\{i|\delta y_{iS}>0\}} \left\{ \frac{x_i^0 - l_i}{\delta y_{is}} \right\} \geq 0, \theta_u = \min_{\{i|\delta y_{iS}<0\}} \left\{ \frac{u_i - x_i^0}{-\delta y_{is}} \right\} \geq 0$$

6 Artificial Variable

Artificial variable is used to find an initial basic solution.

6.1 Big M

Lemma 6.1 (Lower bound for M)

A finite value for such an M must exist, and it is $\max\{c_B^T B^{-1}\}$, here B is the optimal basis for the primal problem.

Proof Suppose B is the optimal basis for the primal problem and x^* is the corresponding optimal solution, it is equal to discuss the problem of introducing multiple new variables and keep the optimality. Thus we need the reduced cost for these new variables being non-negative, that is, $r_{x_a} = M - c_B^T B^{-1} I \geq 0$. ■

6.2 Two phase

7 Transportation Problem

Primal **Dual**

$$\begin{aligned}
\min \quad & \sum_{i,j} c_{ij} x_{ij} & & \max \quad \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = a_i \quad \text{for} \quad i = 1, \ldots, m & & \text{s.t.} \quad u_i + v_j \leqslant c_{ij}, \quad \forall v, j \\
& \sum_{i=1}^{m} x_{ij} = b_j \quad \text{for} \quad j = 1, \ldots, n & & \\
& x_{ij} \geqslant 0 \quad \forall v, j & &
\end{aligned} \tag{10}$$

Bibliography

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